AN ITERATION FORMULA FOR FINDING ROOTS OF EQUATIONS

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We seek the roots of the equation

$$f(x) = 0 \tag{1}$$

The Taylor series expansion of $f(x)$ about a point x_n has the form

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} (x - x_n)^k f^{(k)}(x_n) \tag{2}$$

If we substitute x_{n+1} for x in equation (2) we obtain the series

$$f(x_{n+1}) = \sum_{k=0}^{\infty} \frac{1}{k!} (x_{n+1} - x_n)^k f^{(k)}(x_n) \tag{3}$$

A wide variety of iteration formulae for the numerical solution of problem (1) can be derived from (3) by regarding x_{n+1} as the $(n + 1)$-th iterate in an iterative process determined by truncating (3) after a given number of terms. In this general approach it is assumed that x_{n+1} is a sufficiently accurate approximation to the required root of (1) to justify setting $f(x_{n+1}) = 0$. The method described in this paper involves truncating the series of equation (3) after the term containing $f''(x_n)$ and setting $f(x_{n+1}) = 0$ to arrive at the formula

$$\tfrac{1}{2} (x_{n+1} - x_n)^2 f''(x_n) + (x_{n+1} - x_n) f'(x_n) + f(x_n) = 0. \tag{4}$$

The quadratic factor $(x_{n+1} - x_n)^2$ in this formula can be taken care of by recalling the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

from which we get

$$(x_{n+1} - x_n)^2 = \left[\frac{f(x_n)}{f'(x_n)}\right]^2 \tag{5}$$

(one may choose an appropriate initial guess x_o and apply the Newton-Raphson formula once to obtain the second starting value). To illustrate this we apply the formula in solving our previous problem. As starting values we take $x_o = 2$ and $x_1 = 1.7364865$, the first two iterates we got using the Newton-Raphson formula. The results were

$$x_2 = 1.4647018$$
$$x_3 = 1.4017372$$
$$x_4 = 1.4036010$$
$$x_5 = 1.4036022$$

These results indicate that the modification we have introduced into formula (6) to give formula (7) has not resulted in any significant change in the rate of convergence of the formula. This general remark has been confirmed by results obtained using (7) in solving a large number of other examples.