THE GEOMETRY OF COMMUTATIVE SYMMETRIC LOOPS

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The aim of this paper is to show the connection existing between a commutative symmetric loop and an euclidean space.

Let Q(o) be a locally analytic loop (Mal'cev 1955) with mappings $S{:}x \leftarrow x^{-1}$ and $R{:}x \rightarrow xox$, where x^{-1} is the right inverse of x, $x \in Q$ and the operation (o) is introduced locally. We define $L_x y = xoy$

Definition 1

A locally analytic loop Q(o) is called a symmetric loop (Karanda 1972, 1975) if for all x, y $\in$ Q, the following conditions are fulfilled:

(a) S(xoy) = SxoSy, that is, S is an automorphism.
(b) S^2 = id, that is, S is involutive
(c) $S\ell(x,y) = \ell(x,y)S$, that is, S commutes with

$$\ell(x,y) = L^{-1}_{(xoy)} L_x L_y \qquad \in \alpha S_\ell Q$$

(d) $xo(Sxoy) = y$ or $L_x^{-1} = L_x^{-1} = L_{Sx}$, that is, the loop Q(o) has the left inverse property.
(e) $xo(xoy) = (xox)oy$ or $L_x^2 = L_x{}^2 = L_{Rx}$, that is, the loop Q(o) is left alternative.
(f) The mapping R is locally a bijection.

Definition 2

A quasigroup Q(*) with the identity

$$(x^*y) * (u^*v) = (x^*u) * (y^*v)$$

is called a medial quasigroup, (Beloasov 1969).

Definition 3

A symmetric space (Karanda, 1972, 1975) is a manifold Q with a differential mapping, $\mu{:}Q \times (Q \rightarrow Q$, defined as $\mu(x,y) = x.y = S_x y$

and properties :

1. x. (y.z) = (x.y) . (x.z)
2. x. (x.y) = y
3. x.x = x
4. Every point x has a neighbourhood Ux: x . y = y implies that y = x for all y in Ux.

It is now easy to show the following lemma :

Lemma I

A medial Loos quasigroup Q(.) is abelian.

Note that if Q(o) is a medial symmetric loop, then the mapping R is an automorphism of Q(o) and Q(o) is associative. That is, let x = y and v = u in (1). Then (xox)o(uou) = (xou)o(xou).

By the properties of the symmetric loop, it follows that

RxoRu = R(xou)

and R is an automorphism of Q(o).

And if u = e in (1), then

(xoy)ov = xo(yov),

and thus Q(o) is associative.

Theorem 2

A symmetric loop Q(o) with either the mapping R as an automorphism or the associative law is commutative.

Proof

Let R be an automorphism. Then by the properties of Q(o), it follows that

(RxoRy)oz = xo(Ryo(xoz))

Let z = e, then

RxoRy = xo(Ryox)

But Q(o) is left alternative, therefore

xo(xoRy) = xo(Ryox).

By the left cancellation law, it follows that

xoRy = Ryox,

and Q(o) is commutative.

Q(o) with the associate law is a group and for all x, y ε Q

S(xoy) = SxoSy.

But Q(o) as a symmetric loop has the property

$$S(xoy) = SxoSy.$$

Thus Q(o) is a commutative group.

Theorem 3

A commutative symmetric loop Q(o) is an euclian space (locally).

Proof

Q(o) is an analytic loop and by the Taylor's formula

$$(xoy)^i = f^i(x,y) = x^i + y^i + a^i_{\alpha\beta}\, x^\alpha y^\beta + g^i{}_{\alpha\beta k}\, x^\alpha x^\beta y^k + h^i{}_{\alpha\beta k} x^\alpha y^\beta y^k$$

it follows that, $[x,y]^i = (xoy)^i - (yox)^i = 0$

That is, the structure constants $C^i_{\alpha\beta}$ are equal to zero and the curvature tensor of the space Q(o) vanishes.

Theorem 4

A medial symmetric loop Q(o) with isotopy $xoy = R^{-1}x.Sy$ is an abelian Loos quasigroup Q (.)

Proof

By the given isotopy the medial identity can be changed into

$$S_{R^{-1}(R^{-1}x.Sy)}\, S_{R^{-1}Su} = S_{R^{-1}(R^{-1}x.Su)} S_{R^{-1}Sy} \qquad (2)$$

Let $R^{-1}x = Sy = z$, then $S_{R^{-1}z}\ S_{R^{-1}Su} S_{R^{-1}z} = S_{R^{-1}(z.Su)}$

Inserting this into (2) we have

$$S_{R^{-1}Sy} S_{R^{-1}(R^{-1}x)} S_{R^{-1}Su} = S_{R^{-1}Su}\ S_{R^{-1}(R^{-1}x)} S_{R^{-1}Sy}$$

If

$$R^{-1}(R^{-1}x) = a,\ R^{-1}Sy = b,\ R^{-1}Su = c$$

then Q(.) is abelian, $S_b S_a S_c = S_c S_a S_b$

Theorem 5

(3)

A medial Loos quasigroup Q(.) with isotopy $x\,.\,y = RxoSy$ is a commutative symmetric loop.

Proof

By our isotopy, (3) can be written as

$$RboS(RaoS(RcoSd)) = RcoS(RaoS(RboSd)).$$

If Rb = Sd = e, then SRaoRc RcoSRa, that is Q(o) is a commutative symmetric loop.

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